Pattern Formation in a Compressed Elastic Film on a Compliant Substrate

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ICERM Workshop, Providence, May 30 - June 1, 2012

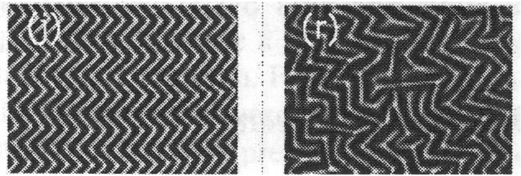
Joint work with Hoai-Minh Nguyen

Funding from NSF is gratefully acknowledged

Outline

Wrinkling of thin films compressed by thick, compliant substrates:

1. Phenomenology: herringbones and labyrinths
2. Energy minimization (using von Karman theory)
3. The energy scaling law (herringbone pattern is optimal)
4. Sketch of the upper bound (the herringbone pattern)
5. Sketch of the lower bound (no other pattern does better)
6. Context (comparison with other examples of wrinkling)

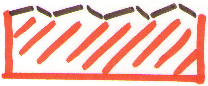


Preprint: on my website math.cims.nyu.edu/faculty/kohn

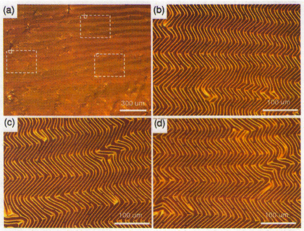
Phenomenology

Wrinkling of thin films compressed by thick, compliant substrates:

- deposit film at high temp then cool; or
- deposit on stretched substrate then release;
- film buckles to avoid compression



Commonly seen pattern: herringbone



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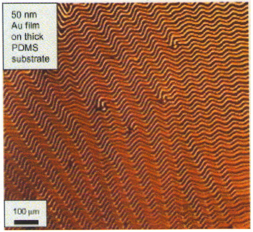

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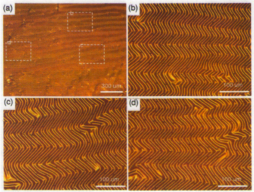
Phenomenology - cont'd

Herringbone pattern when film has some anisotropy, or for specific release histories. Otherwise a less ordered "labyrinth" pattern.



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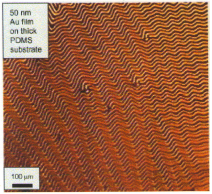
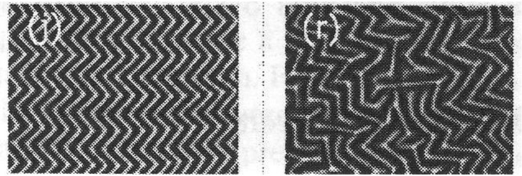
Song et al, *J Appl Phys* 103 (2008) 014303



gold on pdms

Chen & Hutchinson, *Scripta Mat* 50 (2004) 797–801

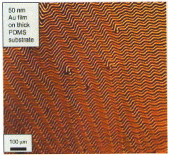


different release histories

Lin & Yang, *Appl Phys Lett* 90 (2007) 241903

The elastic energy

We use a “small-slope” (von Karman) version of elasticity, writing (w_1, w_2, u_3) for the elastic displacement. The energy has three terms:



(1) Membrane energy captures fact that film’s natural length is larger than that of the substrate:

$$\alpha_m h \int |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I|^2 \, dx\, dy$$

(2) Bending energy captures resistance to bending:

$$h^3 \int |\nabla\nabla u_3|^2 \, dx\, dy$$

(3) Substrate energy captures fact that substrate acts as a “spring”, tending to keep film flat:

$$\alpha_s \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right)$$

where $\|g\|^2_{H^{1/2}} = \sum |k||\hat{g}(k)|^2$

The membrane energy

$$E_{\text{membrane}} = \alpha_m h \int |e(w) + \tfrac{1}{2}\nabla u_3 \otimes \nabla u_3 - \eta I|^2 \, dx \, dy$$

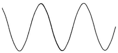
where (w_1, w_2, u_3) is the elastic displacement, and η is the misfit.

- 1D analogue: $\int |\partial_x w_1 + \frac{1}{2}(\partial_x u_3)^2 - \eta|^2 \, dx$
- Explanation: if $(x, 0) \mapsto (x + w_1(x), u_3(x))$ then local stretching is

$$\sqrt{(1 + \partial_x w_1)^2 + (\partial_x u_3)^2} - 1 \approx \partial_x w_1 + \tfrac{1}{2}(\partial_x u_3)^2$$

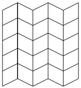
- Vanishes in 1D for sinusoidal profile:

$$w_1 = \eta \frac{\lambda}{4} \sin(4x/\lambda), \;\; u_3 = \sqrt{\eta}\lambda \cos(2x/\lambda)$$



- Our problem is 2D, with isotropic misfit ηI;

  membrane term would vanish for
  piecewise-linear "Miura ori" pattern.



- The herringbone pattern uses sinusoidal wrinkling in two distinct orientations. It does better than the Miura-ori pattern.

The bending energy

$$E_{\text{bending}} = h^3 \int |\nabla\nabla u_3|^2 \, dx \, dy$$

where u_3 is the out-of-plane displacement.

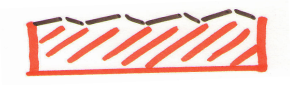
- In a nonlinear theory, bending energy $\sim h^3 \int \kappa_1^2 + \kappa_2^2$ where κ_i are the principal curvatures.
- In our small-slope (von-Karman) setting, the principal curvatures are the eigenvalues of $\nabla\nabla u_3$.

The substrate energy

$$E_{\text{substrate}} = \alpha_s \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right)$$

where (w_1, w_2, u_3) is the (periodic) elastic displacement, and $\|g\|^2_{H^{1/2}} = \sum |k||\hat{g}(k)|^2$.

- Treat substrate as semi-infinite isotropic elastic halfspace.
- Given surface displacement (w_1, w_2, u_3), solve 3D linear elasticity problem in substrate by separation of variables.
- Substrate energy is the result (modulo constants).



Total energy = Membrane + Bending + Substrate

To permit spatial averaging, we assume periodicity on some (large) scale L, and we focus on the energy per unit area:

$$
\begin{aligned}
E_h \quad = \quad & \frac{\alpha_m h}{L^2} \int_{[0,L]^2} \left| e(w) + \tfrac{1}{2} \nabla u_3 \otimes \nabla u_3 - \eta I \right|^2 dx\, dy && \textit{(membrane)} \\
& + \frac{h^3}{L^2} \int_{[0,L]^2} |\nabla \nabla u_3|^2 \, dx\, dy && \textit{(bending)} \\
& + \frac{\alpha_s}{L^2} \left(\|w\|^2_{H^{1/2}} + \|u_3\|^2_{H^{1/2}} \right) && \textit{(substrate)}
\end{aligned}
$$

where h is the thickness of the film.

- We have already normalized by Young's modulus of the film, so α_m, α_s, η are dimensionless parameters:
  - α_m (order 1) comes from justifn of von Karman theory;
  - α_s (small) is the ratio (substrate stiffness)/(film stiffness);
  - η (small) is the misfit.
- Unwrinkled state $(w_1, w_2, u_3) = 0$ has energy $\alpha_m \eta^2 h$.
- Bending term has factor h^3 while membrane term has factor h.

The energy scaling law

Theorem

If h/L and η are small enough, the minimum energy satisfies

$$\min E_h \sim \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\};$$

moreover

- *the first alternative corresponds to the unbuckled state; it is better when $\alpha_m \eta < \alpha_s^{2/3}$.*
- *the second alternative is achieved by a herringbone pattern using wrinkles with length scale $\lambda = \alpha_s^{-1/3} h$, whose direction oscillates on a suitable length scale (not fully determined).*

The smallness conditions are explicit:

$$\alpha_m \alpha_s^{-4/3} (h/L)^2 \leq 1 \quad \text{and} \quad \eta^2 \leq \alpha_m^{-1} \alpha_s^{2/3}.$$

Perhaps other, less-ordered patterns could also be optimal (e.g. "labyrinths"). Numerical results suggest this is the case.

The energy scaling law – cont'd

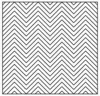
$$\min E_h \sim \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\};$$

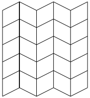
One consequence: the Miura-ori pattern is not optimal:

- Its scaling law is $\alpha_m^{1/6} \alpha_s^{5/8} \eta^{17/16} h$.
- If film prefers to buckle ($\alpha_m \eta \gg \alpha_s^{2/3}$) then Miura-ori energy $\gg$ herringbone energy.

Intuition:

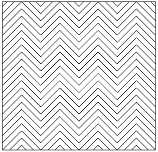
- Bending energy requires folds of Miura-ori pattern to be rounded.
- Where folds intersect this costs significant membrane energy.
- In herringbone pattern the membrane term isn't identically zero, but it does not contribute at leading order.





The herringbone pattern

- Mixture of two symmetry-related "phases"
- "Phase 1" uses sinusoidal wrinkles perp to $(1, 1)$, superimposed on an in-plane shear.
- "Phase 2" uses wrinkles perp to $(1, -1)$, superimposed on a different shear.



In phase 1: $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} \eta & \eta \\ \eta & \eta \end{pmatrix} + \begin{pmatrix} 0 & -\eta \\ -\eta & 0 \end{pmatrix} = \eta I;$

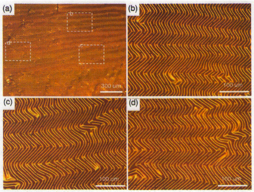
In phase 2: $e(w) + \frac{1}{2}\nabla u_3 \otimes \nabla u_3 = \begin{pmatrix} \eta & -\eta \\ -\eta & \eta \end{pmatrix} + \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix} = \eta I;$

Membrane term vanishes! Since the average in-plane shear is 0, the in-plane displacement w can be periodic.

Herringbone pattern has two length scales:

- The smaller one (the scale of the wrinkling) is set by competition between *bending* term and *substrate energy of* u_3.
- The larger one (scale of the phase mixture) must be s.t. the *substrate energy of w is insignificant*. (It is not fully determined.)

Phenomenology - review



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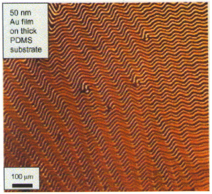
Song et al, *J Appl Phys* 103 (2008) 014303



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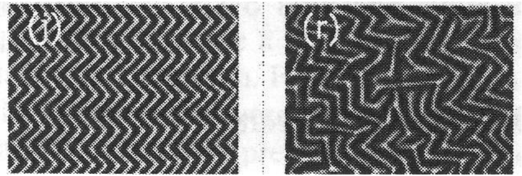

different release histories

Lin & Yang, *Appl Phys Lett* 90 (2007) 241903

No other pattern can do better

Claim: For any periodic (w_1, w_2, u_3), $E_h \geq C \min\{\alpha_m \eta^2 h, \alpha_s^{2/3} \eta h\}$.

Let's take $L = 1$ for simplicity. We'll use only that

$$\text{membrane term} \geq \alpha_m h \int |\partial_x w_1 + \tfrac{1}{2}|\partial_x u_3|^2 - \eta|^2 \, dx \, dy,$$

$$\text{bending term} = h^3 \|\nabla\nabla u_3\|_{L^2}^2, \quad \text{and} \quad \text{substrate term} \geq \alpha_s \|u_3\|_{H^{1/2}}^2.$$

CASE 1: If $\int |\nabla u_3|^2 \ll \eta$ then stretching $\gtrsim \alpha_m \eta^2 h$, since $\partial_x w_1$ has mean 0.

CASE 2: If $\int |\nabla u_3|^2 \gtrsim \eta$ use the interpolation inequality

$$\|\nabla u_3\|_{L^2} \lesssim \|\nabla\nabla u_3\|_{L^2}^{1/3} \|u_3\|_{H^{1/2}}^{2/3}$$

to see that

$$\begin{aligned} \text{Bending + substrate terms} &= h^3 \|\nabla\nabla u_3\|^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 \\ &\gtrsim \left(h^3 \|\nabla\nabla u_3\|^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4 \right)^{1/3} \\ &\gtrsim h\alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h\alpha_s^{2/3} \eta \end{aligned}$$

using arith mean/geom mean inequality.

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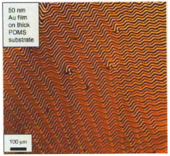
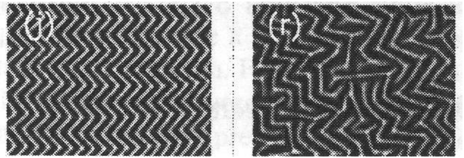
to see that

$$\begin{aligned}
\text{Bending + substrate terms} \quad &= \quad h^3 \|\nabla\nabla u_3\|^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 + \tfrac{1}{2}\alpha_s \|u_3\|_{H^{1/2}}^2 \\
&\gtrsim \quad \left(h^3 \|\nabla\nabla u_3\|^2 \alpha_s^2 \|u_3\|_{H^{1/2}}^4 \right)^{1/3} \\
&\gtrsim \quad h \alpha_s^{2/3} \|\nabla u_3\|_{L^2}^2 \gtrsim h \alpha_s^{2/3} \eta
\end{aligned}$$

using arith mean/geom mean inequality.

Stepping back





Main accomplishment: scaling law of the minimum energy, based on

- upper bound, corresponding to the herringbone pattern, and
- lower bound, using nothing more than interpolation.
- Key point: they have the same scaling law as $h/L \to 0$.

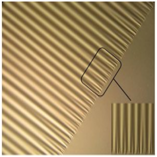
Open questions:

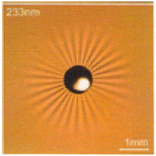
- When is the von Karman model adequate? What changes when the slope of the wrinkling gets large, and/or the strain in the substrate becomes large?
- Can less-ordered patterns achieve the same scaling law?

Context

The membrane energy is nonconvex. It must be (nearly) zero, but this leaves many possibilities. The bending energy is a singular perturbation that acts as a selection mechanism.

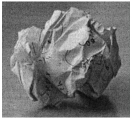
Sometimes tensile effects determine the direction of wrinkling, and the subtle task is to understand its length scale. Examples: wrinkling at the edge of a confined floating sheet (Hoai-Minh Nguyen, today), wrinkling of a hanging drape (Peter Bella and RVK, in preparation), and wrinkling near a drop on a floating sheet (Huang et al).

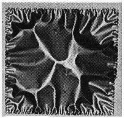


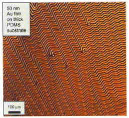


Context – cont'd

When the loads are compressive the geometry has much more freedom. Examples: crumpling of paper, blisters in compressed thin films, and our herringbones.







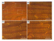
State of the art: we're developing tools by doing examples.

Significant parallels (but also differences) wrt past work on surface energy as a selection mechanism in determining the structure of phase mixtures (e.g. martensitic phase transformation, micromagnetics, and the intermediate state of a type-I superconductor).

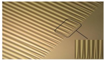
Image credits

P.-C. Lin & S. Yang, *Appl Phys Lett* 90 (2007) 241903

X. Chen and J. Hutchinson, *Scripta Materialia* 50 (2004) 797–801

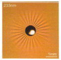


J. Song et al, *J Appl Phys* 103 (2008) 014303

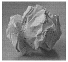


Website of Chris Santangelo, blogs.umass.edu, see also J. Huang et al, *Phys Rev Lett* 105 (2010) 038302

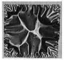
H. Vandeparre et al, *Phys Rev Lett* 106 (2011) 224301



J. Huang et al, *Science* 317 (2007) 650–653



S. Conti and F. Maggi, *Arch Rational Mech Anal* 187 (2008) 1-48



B.-K. Lai et al, *J Power Sources* 195 (2010) 5185-5196